Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Solutions to Assignment 4

Problem 1. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be vectors with Euclidean norm 1. Show that there exist $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$ such that

$$|\epsilon_1 v_1 + \ldots + \epsilon_n v_n| \leq \sqrt{n}.$$

Solution. For each $i \in [n]$, let ε_i be 1 with probability 1/2, and -1 otherwise, independently of other ε_j's.

$$\begin{aligned}
\mathbb{E}[|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n|^2] &= \mathbb{E}\Big[\angle \varepsilon_1 v_1 + \ldots + \varepsilon_n v_n, \varepsilon_1 v_1 + \ldots + \varepsilon_n v_n\Big] \\
&= \mathbb{E}\Big[\sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j \angle v_i, v_j\Big] \\
&= \sum_{1 \leq i,j \leq n} \angle v_i, v_j \mathbb{E}[\varepsilon_i \varepsilon_j] \\
&= \sum_{1 \leq i \leq n} \angle v_i, v_i = n.
\end{aligned}$$

Here the third equality follows by linearity of expectation, and the fourth equality follows as $\mathbb{E}[\varepsilon_i \varepsilon_j] = \mathbb{E}[\varepsilon_i]\mathbb{E}[\varepsilon_j] = 0$ if $i \neq j$ (using the independence of ε_i and ε_j), and $\mathbb{E}[\varepsilon_i^2] = 1$. It follows that for some $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$, we have $|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n|^2 \leq n$, or equivalently $|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| \leq \sqrt{n}$, as required.

Note that if $v_1, \ldots, v_n$ are orthogonal, then $|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| = \sqrt{n}$ for any choice of $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$.

Problem 2. Let H be a bipartite graph with parts A and B such that all vertices in B have degree at most r. Show that there exists a positive constant $c = c(H)$ depending only on H such that any graph G on n vertices with at least $cn^{2-1/r}$ edges contains H as a subgraph.

Solution. Let $|A| = a$, $|B| = b$ and $m = a + b$. Let c be a sufficiently large constant (depending on H) to be determined. Let G be a graph with n vertices and at least $cn^{2-1/r}$

edges. The average degree of G is $d \geq 2cn^{1-1/r}$. Note that

$$\frac{d^r}{n^{r-1}} - \binom{n}{r}\left(\frac{m}{n}\right)^r \geq (2c)^r - m^r \geq a$$

provided that c is sufficiently large depending on a, b and r.

Then, by Lemma 3.2 from lectures, G contains a subset U of at least a vertices such that every r vertices in U have at least $m = a + b$ common neighbours. By Lemma 3.3, H is a subgraph of G.